British Mathematical Olympiad

Round 1 : Wednesday 13th January 1993

Time allowed *Three and a half hours.*

Instructions

- *Full written solutions are required. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1,2,3,4,5 in order.*
- *Staple all the pages neatly together in the top* left *hand corner.*

Do not turn over until **told to do so.**

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1. Find, showing your method, a six-digit integer n with the following properties: (i) n is a perfect square, (ii) the number formed by the last three digits of n is exactly one greater than the number formed by the first three digits of n. (Thus n might look like 123124, although this is not a square.)

2. A square piece of toast $ABCD$ of side length 1 and centre O is cut in half to form two equal pieces ABC and CDA. If the triangle ABC has to be cut into two parts of equal area, one would usually cut along the line of symmetry BO. However, there are other ways of doing this. Find, with justification, the length and location of the shortest straight cut which divides the triangle ABC into two parts of equal area.

3. For each positive integer c, the sequence u_n of integers is defined by

   $u_1 = 1,\ u_2 = c, \quad u_n = (2n+1)u_{n-1} - (n^2-1)u_{n-2},\ (n \geq 3).$

   For which values of c does this sequence have the property that u_i divides u_j whenever $i \leq j$?

   (Note: If x and y are integers, then x *divides* y if and only if there exists an integer z such that $y = xz$. For example, $x = 4$ divides $y = -12$, since we can take $z = -3$.)

4. Two circles touch internally at M. A straight line touches the inner circle at P and cuts the outer circle at Q and R. Prove that $\angle QMP = \angle RMP$.

5. Let x, y, z be positive real numbers satisfying

   $$\frac{1}{3} \leq xy + yz + zx \leq 3.$$

   Determine the range of values for (i) xyz, and (ii) $x + y + z$.